

# Active fluctuation symmetries

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## Abstract:

We elaborate on an observation of Maes-van Wieren (2006) to obtain fluctuation symmetries also for time-symmetric quantities. Examples are given, analytic and numerical, yielding time-symmetric path-observables with fluctuations satisfying a Gallavotti-Cohen type symmetry. From these results one is actually introduced to stationary nonequilibrium by a different phenomenology. It deals with a complementary class of what we may call *active* fluctuation symmetries, again general non-perturbative nonequilibrium relations but not expressed in terms of the traditional dissipative variables; they rather involve the notion of dynamical activity. In particular, we derive Green-Kubo like relations for differences in dynamical activity. The illustrations include boundary driven Kawasaki and zero range models and the spinning Lorentz gas.

## 1 Introduction

General and non-perturbative relations are not so common in nonequilibrium physics. Recent decades have therefore seen a big interest in the fluctuation symmetries of the entropy production as pioneered in the papers [1, 2]. It was found that these symmetries are an expression of local detailed balance, implying that the total path-wise entropy flux is the source term of time-reversal breaking in the nonequilibrium action; see [3, 4, 5, 6, 7]. Local detailed balance refers to the underlying microscopic time-reversibility that governs the contact between the system and each (equilibrium) reservoir in the environment [6, 8, 9, 10, 11, 12]. Also the nonequilibrium free energy relations, called Jarzynski relation after [13], are of a very similar nature.

The natural place to study all these fluctuation relations is in large deviation theory for occupations and currents in nonequilibrium systems. The present paper wants however to take some distance from the original work which concentrated mostly on heat and entropy production. Here we add fluctuation symmetries for aspects of dynamical activity, which belong to the time-symmetric fluctuation sector of a nonequilibrium system. This dynamical activity (or frenesy, as called in the context of linear response [14]) is an important kinetic aspect of nonequilibrium and is of growing importance in fluctuation theory. We refer to [15, 16] for some impressions and further references. The main result of the paper (in Section 4) gives fluctuation symmetries for differences in

dynamical activity.

For the plan of the paper, the next section repeats the central idea after [17, 3, 18, 19]. Besides time-reversal symmetry we add a second symmetry which can be spatial or internal and that gives rise to additional fluctuation symmetries. The standard example of a fluctuation symmetry for the entropy flux is reviewed in Section 3. We mention also how we view the relation with other topics in nonequilibrium statistical mechanics, in particular and less known, how the Kubo formula (and not just the Green-Kubo relations) follow from the fluctuation symmetry. Staying still in the same context we then treat two examples (boundary driven Kawasaki and zero range dynamics) in Section 4 that are explicitly calculated to give fluctuation symmetries for differences in their dynamical activity. There we find the main point of the paper, giving fluctuation–activity relations. Section 5 applies that to the spinning Lorentz gas where the notion of dynamical activity gets a specific physics realization. Computer simulations validate there our guesses also in the non-Gaussian fluctuation sector.

## 2 General observation

We start with the formal content of fluctuation symmetries, [17, 3].

Let  $X \in \Omega$  denote a fluctuating quantity. That means that its outcome is variable and uncertain as, in physical terms, it depends on hidden or more microscopic degrees of freedom. For mathematical modeling we will consider probability distributions for  $X$  where  $\Omega$  is the space of possible outcomes (in the universe). This in turn also means that  $\Omega$  supports some elementary structure such as used for integration. More relevant is the presence of involutions  $\Theta$  and  $\Gamma$  on  $\Omega$ , preserving there such elementary structure, and being mutually commuting,  $\Gamma^2 = \Theta^2 = \text{Id}$ ,  $\Theta\Gamma = \Gamma\Theta$ .

We will always have a reference probability law  $P_o$  for  $X$  which is both  $\Theta$ – and  $\Gamma$ –invariant;  $P_o(\Theta X) = P_o(X) = P_o(\Gamma X)$ . Our main interest is in a probability law  $P$  on  $X$  which we assume has a density with respect to  $P_o$ :

$$dP(X) = e^{-A(X)} dP_o(X) \quad (2.1)$$

for “action”  $A$ , which for our purposes is mostly explicitly known. For all the meaning of (2.1) we pretend for a moment that  $X$  takes a finite number of values so that expectations  $\langle \cdot \rangle$  under  $P$  are simply written as finite sums

$$\langle f(X) \rangle = \sum_x f(x) P(x) = \sum_x f(x) e^{-A(x)} P_o(x)$$

for an observable  $f$  and with  $P(x)$  the probability that  $X = x$ .

### Examples

1. *Dynamical ensembles.*  $\Omega$  can be the set of allowed (possibly coarse-grained) trajectories of a dynamical system. The trajectory  $X$  is then a sequence

of states  $(x_t, t \in [0, T])$  of a (sub)system over a time-interval  $[0, T]$ , and we could take  $\Theta = \text{time-reversal}$ , i.e.,  $\Theta X = (\pi x_{T-t}, t \in [0, T])$  with  $\pi$  the kinematical time-reversal such as flipping the momenta of all particles.

As an illustration of a possible origin of dynamical ensembles we take a collection of sites  $i \in V$  where we think of  $V$  as the volume and to which we assign coupled oscillators  $(q_i, p_i) \in \mathbb{R}^2$ . The dynamics is Hamiltonian with potential  $U$  except at the boundary sites  $i \in \partial V$  where we add Langevin forces:

$$\begin{aligned} dq_i &= p_i dt, \quad i \in V \\ dp_i &= -\frac{\partial U}{\partial q_i}(q) dt, \quad i \in V \setminus \partial V \\ dp_i &= -\frac{\partial U}{\partial q_i}(q) dt - \gamma \kappa_i p_i dt + \sqrt{\frac{2\gamma}{\beta_i}} dW_i(t), \quad i \in \partial V \end{aligned} \quad (2.2)$$

The  $W_i(t)$  are independent standard Brownian motion. When  $\kappa_i \beta_i = \beta$  for all  $i \in \partial V$ , then  $\rho^\beta \propto \exp -\beta\{p^2/2 + U(q)\}$  is a stationary equilibrium distribution for all  $\gamma > 0$ . The stationary process for (2.2) is then reversible, giving the reference  $P_o$  for which  $P_o = P_o \Theta$ . For trajectory  $X = ((p_t, q_t), t \in [0, T])$  we have the time-reversed trajectory  $(\Theta X)_t = (-p_{T-t}, q_{T-t})$ . When  $\kappa_i \equiv 1$  in (2.2) appears the nonequilibrium process  $P$  which can either be started at time zero from  $\rho^\beta$  or from its own stationary density making then a stationary nonequilibrium process. When (always for  $\kappa_i \equiv 1$ ) the inverse temperatures  $\beta_i$  of the reservoirs connected to  $\partial V$  are different, we break time-reversal invariance and also spatially some symmetry will be broken (e.g. between left and right when  $V$  is a linear chain) even when the potential  $U$  is homogeneous. The second symmetry  $\Gamma$  can then for example reflect the volume (reversing left and right, which is  $\Gamma X = (p_t(L-i), q_t(L-i), t \in [0, T], i = 1, 2, \dots, L)$  for  $V = \{0, 1, 2, \dots, L\}, \partial V = \{0, L\}$ . Note of course that  $P(\Gamma X) \neq P(\Theta X)$ . The action  $A$  for (2.1) was calculated in Section 3.1 of [20].

**2. Random matrices** Consider the non-centered Wishart random matrix model; see e.g [21] for the statistic and signal processing literature. Here  $X$  is an  $N \times N$  complex matrix with distribution

$$dP(X) = \frac{1}{Z} e^{-N \text{Tr}[(X-B)^*(X-B)]} dX$$

where  $dX = \prod dX_{ij} d\overline{X_{ij}}$  stands for the volume element in  $\mathbb{R}^{2N^2}$ . The matrix  $B$  is given, and its presence breaks the unitary invariance  $\Theta X = U X U^*$  for a unitary matrix  $U$ . The reference ensemble has  $B = 0$ , with distribution  $P_o$ , so that for (2.1)

$$dP(X) = dP_o(X) e^{N \text{Tr}[B^* X + X^* B] + C}$$

for some normalization  $C$  that depends on  $B$  and  $N$ .

**3. Binomial distribution** Let  $X$  be the number of successes of a repeated and independent Bernoulli experiment with parameter  $a \in [0, 1]$ ; i.e.,  $X =$

$v_1 + v_2 + \dots + v_n$  with  $v_i = 0, 1$  independently with  $\text{Prob}[v_i = 1] = a$ . Take involution  $\Theta X = n - X$ , which leaves invariant the distribution  $P_o$  on  $X$  for  $a = 1/2$ . Then, for  $x = 0, 1, \dots, n$ ,

$$\begin{aligned} P(x) &= P_o(x) 2^n p^x (1-p)^{n-x} = P_o(X) (2(1-p))^n \left(\frac{p}{1-p}\right)^x \\ P(x) &= P(n-x) \left(\frac{p}{1-p}\right)^{2x-n} = P(\Theta X) \exp\{(2x-n) \log \frac{p}{1-p}\} \end{aligned} \quad (2.3)$$

is the distribution of  $X$  for parameter value  $a = p$ .

We continue with our general observation, starting from (2.1). Define

$$\begin{aligned} S &:= A\Theta - A \\ \mathcal{T} &:= A\Theta + A \\ R &:= A\Theta\Gamma - A \end{aligned} \quad (2.4)$$

so that

$$A = \frac{1}{2}(\mathcal{T} - S)$$

Note also that  $2R = \mathcal{T}\Gamma - \mathcal{T} + S + S\Gamma$  so that  $R$  is the antisymmetric part of  $\mathcal{T}$  under  $\Gamma$  when  $S$  is antisymmetric under  $\Gamma$ :

$$S\Gamma = -S \Leftrightarrow R = \frac{1}{2}(\mathcal{T}\Gamma - \mathcal{T}) \quad (2.5)$$

The very definitions (2.4) imply the identities

$$\langle f(\Theta X) \rangle = \sum_x f(x) e^{-A(\Theta x)} P_o(x) = \langle f(X) e^{-S(X)} \rangle \quad (2.6)$$

$$\langle f(\Theta\Gamma X) \rangle = \sum_x f(x) e^{-A(\Theta\Gamma x)} P_o(x) = \langle f(X) e^{-R(X)} \rangle \quad (2.7)$$

for all functions  $f$  on  $\Omega$ . From (2.7) we also have for  $\Theta$ -symmetric observables  $f = f\Theta$  such as  $f = \mathcal{T}\Gamma - \mathcal{T}$  that

$$\langle f\Gamma \rangle = \langle f e^{-\frac{1}{2}(\mathcal{T}\Gamma - \mathcal{T}) - \frac{1}{2}(S + S\Gamma)} \rangle \quad (2.8)$$

which we call an active fluctuation symmetry for reasons that will become clear in Section 4.

There is a rewriting to the more familiar Gallavotti-Cohen type fluctuation symmetries. From (2.6) by taking  $f(x) = \delta(S(x) - \sigma)$  we get

$$\text{Prob}[S(X) = -\sigma] = e^{-\sigma} \text{Prob}[S(X) = \sigma] \quad (2.9)$$

and, from (2.7) by choosing  $f(x) = \delta(R(x) - r)$ ,

$$\text{Prob}[R(X) = -r] = e^{-r} \text{Prob}[R(X) = r] \quad (2.10)$$

with probabilities referring to the probability law  $P$ .

These relations are general and can be applied in a variety of ways. For example, as will be seen in Section 3.2 and in equations (3.14)–(3.16), from (2.6) follows that for all functions  $g$  on  $X$ ,

$$\langle g(X) \rangle - \langle g(\Theta X) \rangle = \langle g(X) S(X) \rangle^\circ$$

to first order in the action  $A$  of (2.1) and where the last expectation  $\langle \cdot \rangle^\circ$  is with respect to  $P_o$ . That is a (generalized) Kubo or fluctuation–dissipation relation, valid around equilibrium  $P_o$ . Higher orders and nonlinear response around equilibrium can be obtained from combining (2.6) and (2.7) as done in [22]. But the same holds replacing  $\Theta \rightarrow \Theta\Gamma$  and  $S \rightarrow R$  in which case we will arrive at fluctuation–activity relations [23].

Another easy consequence is that always  $\langle R(X) \rangle \geq 0$ ,  $\langle \mathcal{T}\Gamma - \mathcal{T} \rangle \geq 0$  and  $\langle S(X) \rangle \geq 0$ , for example useful to determine the direction of currents, [19].

All these consequences remain basically intact also for variables that differ from  $S$  or  $R$  by a total (time-)difference as long as some boundedness of these terms can be assured (referring to the first example above). We then get asymptotic fluctuation symmetries, where (2.6)–(2.10) are not exact but only valid in some limit (of observation time).

It goes without saying that the relevance of the fluctuation identities (2.6)–(2.7) depends crucially on the systematic and operational meaning of  $S$  and  $\mathcal{T}$ . It was understood before that  $S$  is deeply related to changes in entropy (as we will repeat in the next Section), and Sections 4–5 treat examples where  $\mathcal{T}$  is made visible and related to the dynamical activity.

### 3 Standard example: entropy flux

The present section contains the standard application of (2.6) to obtain a fluctuation symmetry for the total entropy flux in a model of nonequilibrium. There will be nothing particularly new here, except for the style of presentation, with its emphasis on (2.6) and some reflections towards the end of the section connecting the fluctuation symmetry also with response theory. Examples for spatially extended systems are not so common in the literature on fluctuation symmetries and Section 4 will provide some.

To be more specific, consider a Markov jump process on a finite state space  $K$ . We specify the transition rates  $k_t(x, y)$  (time-dependent) for jumps  $x \rightarrow y$  as

$$k_t(x, y) = \psi(x, y) \exp\left\{\frac{\beta_t}{2}[U(x, a_t) - U(y, a_t) + F(x, y)]\right\} \quad (3.1)$$

where  $a_t$  is a time-dependent (external) protocol changing the energy function  $U$ . We take the driving  $F(x, y) = -F(y, x)$  antisymmetric and the reactivities  $\psi(x, y) = \psi(y, x)$  symmetric. The other time-dependent parameter  $\beta_t \geq 0$  is the changing inverse temperature of the environment (in units where  $k_B = 1$ );

the nonequilibrium driving sits entirely in the function  $F$  and in the time-dependence of both the protocol  $a_t$  and the inverse temperature  $\beta_t$ . We call  $U(x, a)$  the energy of the system when in state  $x$  for external value  $a$ , because we imagine that the changes in  $U$  are exactly balanced by the change of energy in the environment. Clearly, if  $F = 0$  and when  $a_t = a, \beta_t = \beta$  are constant, then the process is reversible with stationary distribution  $\rho^\beta(x) \propto \exp -\beta U(x, a)$ .

For a given path  $X = (x_t, t \in [0, T])$  over the time-interval  $[0, T]$  the energy changes

$$U(x_T, a_T) - U(x_0, a_0) = \sum_{s \leq T} [U(x_s, a_s) - U(x_{s-}, a_s)] + \int_0^T \frac{\partial U}{\partial a_t}(x_t, a_t) \dot{a}_t dt \quad (3.2)$$

because of two effects: for fixed value  $a_t$  the state changes and energy is exchanged with the environment as heat

$$Q_o(X) := \sum_{s \leq T} [U(x_s, a_s) - U(x_{s-}, a_s)] \quad (3.3)$$

(sum over jump times in  $X$ ). Secondly, for fixed state  $x_t$  the external value changes  $\dot{a}_t = \frac{da_t}{dt}$ , doing work

$$W_o(X) := \int_0^T \frac{\partial U}{\partial a_t}(x_t, a_t) \dot{a}_t dt$$

Equation (3.2) mimics the first law of thermodynamics. The change in energy of the system equals the change in internal energy received as heat  $Q_o$  from the environment plus the amount of work  $W_o$  done on the system by the environment:

$$U(x_T, a_T) - U(x_0, a_0) = Q_o(X) + W_o(X)$$

The nonequilibrium driving  $F$  can be added and subtracted from that balance. We think of it as doing work on the system, which is instantaneously released as heat, so that now  $U(x_T, a_T) - U(x_0, a_0) = Q(X) + W(X)$ , but with

$$Q(X) := Q_o(X) - \sum_t F(x_{t-}, x_t), \quad W(X) := W_o(X) + \sum_t F(x_{t-}, x_t) \quad (3.4)$$

all depending on a specific path  $X$ . We refer to [24] for more details and insights on stochastic energetics.

In the same spirit we can also associate a change in entropy of the environment to a trajectory  $X$ . The idea is that the environment consists of big equilibrium reservoirs undergoing only reversible changes in interaction with the system. We look back at (3.3) and (3.4) to define

$$S_{\text{OUT}}(X) := - \sum_s \beta_s \delta Q_s = \sum_s \beta_s \{F(x_{s-}, x_s) - [U(x_s, a_s) - U(x_{s-}, a_s)]\} \quad (3.5)$$

for the change of the entropy in the environment (always per  $k_B$ ). That is called the entropy flux, which can be split into a reversible part, due to the energy exchange, and an irreversible part

$$\sigma(X) := \sum_t \beta_t F(x_{t-}, x_t) \quad (3.6)$$

We now repeat the observation of [3, 6] that the entropy flux (3.5) can be obtained as for (2.6) as the source term of time-reversal breaking.

For a good moment let us leave out the kinematical time-reversal  $\pi$  on  $K$  and proceed with the undecorated time-reversal  $\Theta$  defined on paths  $X$  via  $(\Theta X)_t = X_{T-t}$  for  $t \in [0, T]$ . We check from (3.5) that  $S_{\text{OUT}}(X) = -S_{\text{OUT}}(\Theta X)$ , or, the entropy flux per path is antisymmetric under time-reversal. Let now  $P_\mu$  be the path distribution when we start at time zero from the law  $\mu$ . The time-dependence of the protocol can be reversed to define  $\tilde{k}_t(x, y) := k_{T-t}(x, y)$ . Choosing a law  $\nu$  on  $K$  the latter Markov process has a distribution  $\tilde{P}_\nu$  on the paths  $X = (x_t, t \in [0, T])$ . Assuming  $\mu, \nu > 0$  and (the dynamical reversibility) that  $k_t(x, y) = 0$  implies  $k_t(y, x) = 0$ , we can find the  $S$  in (2.6) by writing

$$\frac{dP_\mu}{d\tilde{P}_\nu \Theta} = e^S \quad (3.7)$$

and get

$$S(X) = \log \frac{\mu(x_0)}{\nu(x_T)} + \frac{k_{t_1}(x_0, x_{t_1})k_{t_2}(x_{t_1}, x_{t_2}) \dots k_{t_n}(x_{t_{n-1}}, x_T)}{k_{t_n}(x_T, x_{t_{n-1}}) \dots k_{t_1}(x_{t_2}, x_{t_1})k_{t_1}(x_{t_1}, x_0)}$$

for jump times  $t_1, t_2, \dots, t_n$  in  $X$ . Indeed the jump times in  $\Theta X$  are respectively  $T - t_n, \dots, T - t_2, T - t_1$ , and we have substituted  $\tilde{k}_{T-t_n}(x_T, x_{t_{n-1}}) = k_{t_n}(x_T, x_{t_{n-1}})$  etc. One can see what this becomes for (3.1). Substituting into the previous formula makes

$$S(X) - \log \frac{\mu(x_0)}{\nu(x_T)} = \sum_t \beta_t \{U(x_{t-}, a_t) - U(x_t, a_t) + F(x_{t-}, x_t)\} \quad (3.8)$$

which is (3.5). That relation can be called a (generalized) Crooks relation [5], and for  $F \equiv 0$  it almost immediately produces Jarzynski identities which are used to evaluate equilibrium free energies from the fluctuations of the dissipative work — we refer to the literature and the references therein for more details, [13, 25, 7].

Let us now specify to the case where  $\beta_t = \beta, a_t = a$  are constant in time. In particular, with respect to (3.6), and for state functions  $h_\mu(x) := \log \mu(x) + \beta U(x), h_\nu(x) := \log \nu(x) + \beta U(x)$ , we have the identity

$$S(X) = \beta \sum_t F(x_{t-}, x_t) + h_\mu(x_0) - h_\nu(x_T) \quad (3.9)$$

for all trajectories  $X$ . Note that the left-hand side is defined from (3.7) implementing (2.6), while the right-hand side is defined from the heat and (3.5)–(3.6). Therefore, the identities (3.8)–(3.9) are the core of what is generally called the fluctuation symmetry, the fluctuation relations or the fluctuation theorem (transient or steady state) for the entropy production [7].

### 3.1 Exact fluctuation symmetry

In the following we restrict ourselves to time-homogeneous Markov processes and we do no longer write the dependence on  $a_t = a$ . We take also  $\beta = 1$ .

Consider the reference reversible process  $P_o$  started in equilibrium  $\rho_o$  for which there is detailed balance with rates

$$k_o(x, y) = \psi(x, y) e^{\frac{1}{2}[U(x) - U(y)]}, \quad \rho_o(x) = \frac{1}{Z} e^{-U(x)}.$$

The nonequilibrium process has rates  $k(x, y) = k_o(x, y) \exp F(x, y)/2$  and we choose to start it also from  $\rho_o$ . Its distribution on paths  $X$  in the time-interval  $[0, T]$  is then denoted by  $P$ . We proceed as in (2.4) to find

$$S(X) = \sum_t F(x_{t-}, x_t), \quad \mathcal{T}(X) = 2 \int_0^T [\xi(x_s) - \xi_o(x_s)] ds \quad (3.10)$$

for escape rates  $\xi(x) := \sum_y k(x, y)$ . Now clearly (2.6) holds, and with  $f(X) = \exp[-zS(X)]$  for all  $z \in \mathbb{C}$ , we have the exact fluctuation symmetry

$$\langle e^{-zS(X)} \rangle = \langle e^{-(1-z)S(X)} \rangle \quad (3.11)$$

with expectations in the nonequilibrium process starting from the equilibrium distribution  $\rho_o$ .

Another way to get an exact fluctuation symmetry is to look back at (3.9) with probabilities  $\nu = \mu = \rho$  equal to the stationary distribution of the nonequilibrium process. We then have from (3.7) when combined with (3.9) that in the nonequilibrium steady regime, for all  $T$ ,

$$\langle f(X) \rangle = \langle e^{-\sigma(X) - h(x_0) + h(x_T)} f(\Theta X) \rangle \quad (3.12)$$

for irreversible entropy flux  $\sigma(X) = \beta \sum_t F(x_{t-}, x_t)$  and state function  $h(x) := \log \rho(x) + \beta U(x)$ . The exact symmetry (3.12) invites to give special physical meaning also to that function  $h$ , but there is no convincing thermodynamic or operational meaning yet. That is also why asymptotic (in  $T \uparrow +\infty$ ) fluctuation symmetries have been more appreciated, obtained from (3.12) for  $f$  any positive function of  $\sigma(X)$ , by taking the logarithm on both sides and using the boundedness of the function  $h$  which makes it disappear when finally dividing by  $T$  and letting  $T \uparrow +\infty$ .



### 3.2 Relations with other aspects of nonequilibria

The above techniques and relations are not new. Looking backward, it appears that their main input has been the relation (3.8). That has analogues for diffusion process [4, 26, 27], for dynamical systems [2, 17, 28, 29] and also for non-Markovian processes [3, 30, 31] as long as there is sufficient space-time locality to ensure a large deviation principle [3]. The main origin of the fluctuation symmetry is therefore the identification of the entropy flux as marker of time-reversal breaking, [17, 3, 5, 6]. We note next some other relations with aspects of nonequilibrium statistical mechanics.

Quite some features of the close-to-equilibrium regime are easily deduced from the fluctuation symmetry. There are for example the Green-Kubo relations, with Onsager reciprocity as first explained in [32] following from an extended fluctuation symmetry. The fluctuation-dissipation theorem with the Kubo formula [33] is a more general consequence. More globally, the validity of the McLennan ensemble close-to-equilibrium is another implication, see [34, 35].

We will illustrate just one aspect which we have not seen stated as such, and which is useful. Start again from (2.6) and take a function  $f(X) = g(\Theta X) - g(X)$  in terms of another function  $g$  of interest. Then,

$$\langle g(X) \rangle = \langle g(\Theta X) \rangle + \langle (g(\Theta X) - g(X)) e^{-S(X)} \rangle \quad (3.13)$$

Imagine now that the action  $A$  in (2.1) is small, so that the law  $P$  is just a small perturbation of the reference law  $P_o$  and so that  $S = A\Theta - A$  is small. We can then expand the last term in (3.13) to bring

$$\begin{aligned} \langle g(X) \rangle &= \langle g(\Theta X) \rangle + \langle g(\Theta X) - g(X) \rangle - \langle (g(\Theta X) - g(X)) S(X) \rangle^o \\ &= \langle g(\Theta X) \rangle + \langle g(X) S(X) \rangle^o \end{aligned} \quad (3.14)$$

where the last expectation, with the superscript  $\langle \cdot \rangle^o$ , is with respect to the reference  $P_o$  and we have used that  $P_o$  is  $\Theta$ -invariant. That linear order relation can be applied to the context of dynamical ensembles as we had it above, with  $\Theta$  time-reversal on trajectories  $X = (x_t, t \in [0, T])$ . Take for example  $g(X) = O(x_T)$  so that  $g(\Theta X) = O(x_0)$  for a state function  $O$ . We then obtain from (3.14) the Kubo formula

$$\langle O(x_T) \rangle = \langle O(x_0) \rangle + \langle O(x_T) S(X) \rangle_o^{eq} \quad (3.15)$$

where the expectations refer to the process  $P$  started from equilibrium  $\rho_o$  at time zero. Indeed, we should substitute in (3.15) the expression (3.8) for  $S(X)$  with  $F \equiv 0$ ,  $\beta_t \equiv \beta$ ,  $a_t = a - \varepsilon_t \theta(t)$  and  $\mu = \nu = \rho_o$  being the equilibrium distribution with potential  $U(x, a)$ . Then, still using the first law (3.2), we arrive at the more familiar linear response expression

$$\begin{aligned} \langle O(x_T) \rangle - \langle O(x_0) \rangle_o^{eq} &= \\ \langle O(x_T) S(X) \rangle_o^{eq} &= \int_0^T ds \varepsilon_s \frac{d}{ds} \langle O(x_t) \frac{\partial}{\partial a} U(x_s, a) \rangle_o^{eq} \end{aligned} \quad (3.16)$$

Yet, it takes the combination (3.8)–(3.15) to immediately understand why this formula is truthfully called fluctuation-*dissipation* relation.

Moving beyond the linear response around equilibrium, makes it more difficult to find specific consequences. Of course, the fluctuation relations hold unperturbed but there is no direct way to derive more specific results. In fact, it appears that one really needs more information about the time-symmetric part,  $\mathcal{T}$  in (2.4), to move further, [22, 36], and that is also part of the motivation of the next sections.

## 4 Symmetry in dynamical activity

We come to give examples of the fluctuation symmetry (2.7), referred to in the title of the paper as *active* because they deal with the dynamical activity.

### 4.1 Boundary driven Kawasaki dynamics

Take  $K = \{0, 1\}^{\{1, 2, \dots, L\}}$ , where states are particle configurations  $x = (x(i), i \in \{1, 2, \dots, L\})$ ,  $x(i) = 0, 1$ , interpreted as vacant *versus* occupied sites on a lattice interval. In other words we are speaking about indistinguishable particles subject to exclusion on a lattice interval. The dynamics has two parts. First, bulk exchange of neighboring occupations, for inverse temperature  $\beta \geq 0$ ,

$$k(x, y) = \exp -\frac{\beta}{2}[V(y) - V(x)]$$

when  $y(j) = x(j)$  for all  $j$  except for  $y(i) = x(i+1)$ ,  $y(i+1) = x(i)$  for some  $i = 1, 2, \dots, L-1$ . The interaction between neighboring sites is ruled by the potential

$$U(x) = -\kappa \sum_{i=1}^{L-1} x(i)x(i+1)$$

where  $\kappa \in \mathbb{R}$  is the coupling parameter. Apart from that interacting diffusion part to the dynamics, there are also the reactions at the boundary sites; there is creation and annihilation of particles at  $i = 1$  and  $i = L$ , with rates

$$k(x, y) = \exp -\frac{\beta}{2}[U(y) - U(x)] \exp \frac{1}{2}A(x, y)$$

for  $y(j) = x(j)$  except for  $j = 1$  where  $y(1) = 1 - x(1)$ , or for  $y(j) = x(j)$  except for  $j = L$  where  $y(L) = 1 - x(L)$ , and

$$\begin{aligned} A(x, y) &= +(a + \delta) \text{ when } y(1) = 1, x(1) = 0, y(j) = x(j), j \neq 1, \\ &= -(a + \delta) \text{ when } y(1) = 0, x(1) = 1, y(j) = x(j), j \neq 1, \\ &= +(a - \delta) \text{ when } y(L) = 1, x(L) = 0, y(j) = x(j), j \neq L, \\ &= -(a - \delta) \text{ when } y(L) = 0, x(L) = 1, y(j) = x(j), j \neq L \end{aligned} \quad (4.1)$$

for some fixed parameters  $a, \delta \in \mathbb{R}$ . The interpretation is that there are left and right chemical potentials  $\mu_\ell, \mu_r$  of the particle reservoirs of the interval with  $\mu_\ell \beta := a + \delta, \mu_r \beta := a - \delta$ . For all other transitions  $k(x, y) = 0$ . As a result,

$$k(x, y) = k_o(x, y) \exp\left[\frac{\delta}{2} J(x, y)\right]$$

with  $k_o(x, y) = \exp[\mathcal{S}(y) - \mathcal{S}(x)]/2$ ,  $\mathcal{S}(x) := -\beta U(x) + aN(x)$ ,  $N(x) := \sum_{i=1}^L x(i)$  (number of particles in the system for state  $x$ ), and *current*

$$\begin{aligned} J(x, y) &= +1 \text{ when a particle enters at } i = 1 \\ &= -1 \text{ when a particle leaves at } i = 1 \\ &= +1 \text{ when a particle leaves at } i = L \\ &= -1 \text{ when a particle enters at } i = L \end{aligned} \quad (4.2)$$

and zero otherwise. In other words,  $J(x, y) = J_r(x, y) - J_\ell(x, y)$  with  $J_\ell(x, y)$  the current of particles into the left reservoir under the transition  $x \rightarrow y$ , and  $J_r(x, y)$  the current of particles into the right reservoir.

For  $\delta = 0$  (and only for  $\delta = 0$ ) there is detailed balance with grand-canonical ensemble

$$\rho_o(x) = \frac{1}{Z} \exp \mathcal{S}(x).$$

Then,  $a/\beta$  is the chemical potential of both particle reservoirs left and right. That equilibrium process determines our reference distribution  $P_o$ . Nonequilibrium arises from taking  $\delta \neq 0$ , which makes the chemical potentials in the imagined left and right particle reservoirs different. We can start the nonequilibrium process from the same  $\rho_o$ , giving our distribution  $P$ , but asymptotically in time a nonequilibrium steady regime will develop. In particular it can be proven that for  $\delta > 0$  there will be a steady particle current from left to right. See for example [37] for the details of the standard fluctuation symmetry as in the previous section.

The decomposition (2.4) here gives

$$S(X) = \delta [J_\ell(X) - J_r(X)] \quad (4.3)$$

with  $S(\Theta X) = -S(X)$  for  $\Theta$  time-reversal, and  $J_\ell(X) := \sum_t J_\ell(x_{t-}, x_t)$  the net number of particles that escape from the interval to the left particle reservoir. Note that  $J_r(X) + J_\ell(X) = -\mathcal{N}(x_T) + \mathcal{N}(x_0)$ , the change of the number of particles in the system.

For the time-symmetric part of the action we have computed from (3.10) that

$$\mathcal{T}(X) = 2 \int_0^T dt [B_1(x_t; a, \delta) + B_L(x_t; a, \delta)] \quad (4.4)$$

where

$$\begin{aligned}
B_1(x; a, \delta) &:= e^{(a+\delta)/2} - e^{a/2} + \{e^{-(a+\delta)/2} - e^{(a+\delta)/2} + e^{a/2} - e^{-a/2}\}x(1) \\
&+ (e^{(a+\delta)/2} - e^{a/2})(e^{\beta\kappa/2} - 1)x(2) \\
&+ \{(e^{-\beta\kappa/2} - 1)(e^{-(a+\delta)/2} - e^{-a/2}) \\
&- (e^{\beta\kappa/2} - 1)(e^{(a+\delta)/2} - e^{a/2})\}x(1)x(2)
\end{aligned}$$

and

$$\begin{aligned}
B_L(x; a, \delta) &:= e^{(a-\delta)/2} - e^{a/2} + \{e^{-(a-\delta)/2} - e^{(a-\delta)/2} + e^{a/2} - e^{-a/2}\}x(L) \\
&+ (e^{(a-\delta)/2} - e^{a/2})(e^{\beta\kappa/2} - 1)x(L-1) \\
&+ \{(e^{-\beta\kappa/2} - 1)(e^{-(a-\delta)/2} - e^{-a/2}) - (e^{\beta\kappa/2} - 1)(e^{(a-\delta)/2} \\
&- e^{a/2})\}x(L-1)x(L)
\end{aligned}$$

We next apply the mirror symmetry  $\Gamma$  through which  $(\Gamma X)_t(i) = x_t(L-i+1)$ . Observe that in that mirror  $J_\ell(X) = J_r(\Gamma X)$ ,  $S\Gamma = -S$ . We can thus compute

$$R(X) = \frac{1}{2}(\mathcal{T}(\Gamma X) - \mathcal{T}(X)) = \int_0^T dt r(x_t) \quad (4.5)$$

from the expected difference in transitions (jumps in and out of the system) left *versus* right, to find

$$\begin{aligned}
r(x) &= -2 \sinh \frac{\delta}{2} ((e^{-\beta\kappa/2} - 1)e^{-a/2} + (e^{\beta\kappa/2} - 1)e^{a/2})(x(L)x(L-1) - x(1)x(2)) \\
&+ 2 \left( \sinh \frac{a-\delta}{2} - \sinh \frac{a+\delta}{2} \right) (x(L) - x(1)) \\
&+ 2e^{a/2} \sinh \frac{\delta}{2} (e^{\beta\kappa/2} - 1)(x(L-1) - x(2))
\end{aligned} \quad (4.6)$$

which is of course also odd in the driving field  $\delta$ . For the boundary driven symmetric exclusion process we must take the coupling  $\kappa = 0$ , and only survives the term

$$r^{\kappa=0} = 2 \left( \sinh \frac{a-\delta}{2} - \sinh \frac{a+\delta}{2} \right) (x(L) - x(1)), \quad (4.7)$$

given entirely in terms of the difference in occupations at the outer sites.

It follows from the general analysis in Section 2 that  $R(X)$  in (4.5) verifies the fluctuation symmetries (2.7)–(2.10), which is a non-trivial general identity whose meaning refers to the reflection-antisymmetric part in the dynamical activity (4.4). In particular, that identity (2.7) for that same  $R$  in (4.5)–(4.6) remains strictly valid even when modifying the interaction potential  $U$  in the bulk of the system. On the other hand, applying the general consequence that  $\langle R(X) \rangle \geq 0$ , or  $\sum_x r(x) \rho(x) \geq 0$ , to (4.7) only gives the well known fact that the density is larger at the side of the largest chemical potential.

## 4.2 Boundary driven zero range process

We discuss next the application of fluctuation symmetries to a bosonic version of the previous example, where particles diffuse without exclusion principle. Consider again a one-dimensional channel composed of  $L$  cells in which we observe occupation numbers  $n(k) \in \mathbb{N}$ ,  $k = 1, \dots, L$ . The particle configuration  $x = (n(1), \dots, n(L))$  can change in two ways. First, via bulk hopping  $x \rightarrow x - e_i + e_{i\pm 1}$  at a rate  $w(n(i))$ , where  $e_i$  stands for the particle configuration with one particle in cell  $i$  and zero elsewhere. The choice  $w(n(i)) \propto n(i)$  corresponds to independent particles. Secondly, at the boundaries, the channel is connected to particle reservoirs with chemical potentials  $\mu_1 = \tilde{\mu}$ ,  $\mu_L = \mu$ , respectively. The transition rates for the creation/annihilation of particles at the boundary sites are then

$$\begin{aligned} k(x, x - e_1) &= s_1 w(n_1) \\ k(x, x + e_1) &= \tilde{r}_1 := r_1 e^\delta \\ k(x, x - e_L) &= s_L w(n_L) \\ k(x, x + e_L) &= \tilde{r}_L := r_L e^{-\delta} \end{aligned} \quad (4.8)$$

The rates for these transitions evoke the chemical potentials at the boundary walls from  $\mu = \log(\tilde{r}_L/s_L)$  and  $\tilde{\mu} = \log(\tilde{r}_1/s_1)$ . We assume that  $s_1/r_1 = s_L/r_L$  so that, for  $\delta = 0$ , we have the equilibrium situation where the chemical potentials left and right become equal. Of course we could have chosen also to modify the  $s_1, s_L$ , but it appears physically most accessible to change the incoming rates  $r_1 \rightarrow \tilde{r}_1, r_L \rightarrow \tilde{r}_L$  to achieve a nonequilibrium regime as we also do in the next section. In fact, to make the equilibrium left/right symmetric we also take  $s_1 = s_L, r_1 = r_L$ .

The corresponding stationary distributions  $\rho_o$  (at  $\delta = 0$ ) and  $\rho$  (at general  $\delta$ ) are product distributions but that will not be used in the following.

We consider trajectories  $X = (x_t, t \in [0, T])$ . Both the equilibrium  $P_o$  and the nonequilibrium process  $P$  start from the same equilibrium distribution  $\rho_o$ . The action (2.1) is easily calculated to be

$$A(X) = \log \frac{r_1}{\tilde{r}_1} I_-^\ell(X) + \log \frac{r_L}{\tilde{r}_L} I_-^r(X) - T (r_1 + r_L - \tilde{r}_1 - \tilde{r}_L) \quad (4.9)$$

where e.g.  $I_-^\ell(X)$  indicates the number of particles entering the system from the left reservoir for the path  $X$ . As we apply time-reversal  $\Theta$ , we obtain the time anti-symmetric part of the action  $S(X) = A(\Theta X) - A(X)$

$$\begin{aligned} S &= \log \frac{r_1}{\tilde{r}_1} (I_-^\ell - I_+^\ell) + \log \frac{r_L}{\tilde{r}_L} (I_-^r - I_+^r) \\ &= \delta (J^r - J_\ell) \end{aligned} \quad (4.10)$$

where e.g.  $J_\ell := I_-^\ell - I_+^\ell$  is the net number of particles that have escaped to the left particle reservoir during  $[0, T]$ . As usual and as explained before,

that entropy production satisfies the exact fluctuation symmetry (2.9). For the asymptotic form, one must be more careful because of the unbounded number of particles; see [38]. Here we are however more interested in the dynamical activity.

That is given in the time-symmetric term  $\mathcal{T}(X) = A(\Theta X) + A(X)$ ,

$$\mathcal{T} = -\delta (I_{\rightarrow}^{\ell} + I_{\leftarrow}^{\ell}) + \delta (I_{\leftarrow}^r + I_{\rightarrow}^r) - 2(\tilde{r}_1 + \tilde{r}_L - r_L - r_1) T \quad (4.11)$$

which is the analogue to (4.4). As there, we now apply the mirror transformation  $\Gamma$ , reversing left/right. First note that again  $S$  is antisymmetric under  $\Gamma$ ,  $S\Gamma = -S$ . On the other hand, we have

$$\mathcal{T}(\Gamma X) - \mathcal{T}(X) = 2\delta (I_{\rightarrow}^{\ell} + I_{\leftarrow}^{\ell} - I_{\rightarrow}^r - I_{\leftarrow}^r) \quad (4.12)$$

exactly proportional to the difference in dynamical activity

$$\Delta(X) := I_{\rightarrow}^r + I_{\leftarrow}^r - I_{\rightarrow}^{\ell} - I_{\leftarrow}^{\ell}$$

between the right and left boundary. Following the logic of (2.7), that suffices for  $\mathcal{T}\Gamma - \mathcal{T} \propto \Delta$  to satisfy a fluctuation symmetry (2.10) up to a total time-difference: when  $f\Theta = f$  is time-symmetric, then

$$\langle f(\Gamma X) \rangle = \langle f(X) e^{\delta \Delta(X)} \rangle \quad (4.13)$$

for all times  $T$ , where we start the nonequilibrium process at time zero from  $\rho_o$ . For example taking  $f = \Delta$ , to first order in  $\delta$ ,

$$\langle \Delta(X) \rangle = -\frac{\delta}{2} \langle \Delta^2(X) \rangle_0^{eq} \quad (4.14)$$

which is formally similar to a Green-Kubo relation but the observable  $\Delta$  is time-symmetric.

It is in fact true for all  $\delta \geq 0$  that  $\langle \Delta \rangle \leq 0$  which means that the greatest activity is to be found at the boundary side of the largest chemical potential. In other words, as for the boundary driven Kawasaki dynamics also for zero range, the particle current can be said to be directed away from the region of largest activity. These statements all hold for any form of the bulk rate  $w$  and are quite independent of the usual statements involving the fluctuation symmetry of entropy production or of currents.

## 5 Spinning Lorentz Gas

The Spinning Lorentz gas (SLG) is a classical mechanical model of particle scattering in 2D; it is actually an interacting version of the normal Lorentz gas [39], which is a well known example of deterministic particle diffusion [40, 41]. The SLG has the additional feature of providing local thermalization of the wandering particles along with the scatterers; a complete description of this

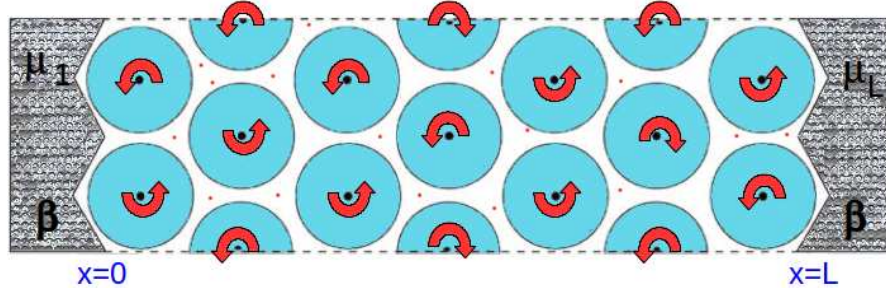


Figure 1: In the Spinning Lorentz Gas (SLG)  $M$  disks with radius one and center fixed in a triangular lattice rotate freely and exchange energy with particles (of mass one) via elastic collisions [42]. The particles move mechanically inside the slab of length  $L$  with periodic boundary conditions in the vertical coordinate. The slab is placed among thermo-chemical reservoirs (ideal gases) with (for the present paper) equal inverse temperatures  $\beta$  and different chemical potentials  $\mu_{i=1,L}$ . The particles can enter and leave to/from the reservoirs at the left and right boundaries.

and the coupled energy and mass transport properties of the SLG model can be found in [42]. As a matter of fact, the validity of the fluctuation theorem for the entropy production and for the joint distribution of currents has been tested for this model before, of course taking into account the limitations due to the unbounded kinetic energy, see [43]. Also, a precise meaning to the state function  $h$  of (3.12) can be found in the SLG context, for the exact fluctuation symmetry case [43]. Specifically, by including such model here we aim to aid extending the study of the nonequilibrium fluctuation relations within the realm of the time-symmetric variables (the dynamical activity). For this purpose we use closely the relations obtained for the previous example in Section 4.2 to apply them in this somewhat more realistic transport model.

As illustrated in Figure 1, in the SLG the array of scatterers is connected to thermo-chemical reservoirs, with nominal chemical potential  $\mu_i, i = 1, L$  and with inverse temperatures  $\beta$ . This setting drives the system into a nonequilibrium stationary regime, when the chemical potentials of the reservoirs are different.

The SLG model is of course more mechanical than the boundary driven zero range model of the previous section. We want to connect them however. At the walls, the particles can enter and can leave the system. The rates at which particles enter are in general related to the mean density  $u$  of their reservoir as  $\propto \frac{u}{\sqrt{\beta}}$ ; see also [44]. In the nonequilibrium setting, there is a reservoir chemical potential difference, given by  $\beta\Delta\mu = \beta(\mu_L - \mu_1) = \log(u_L/u_1)$ ; and hence, in the notation of the previous section, we put  $\tilde{r}_L = \tilde{r}_1 e^{\beta\Delta\mu}$ , or  $2\delta = -\beta\Delta\mu$ .

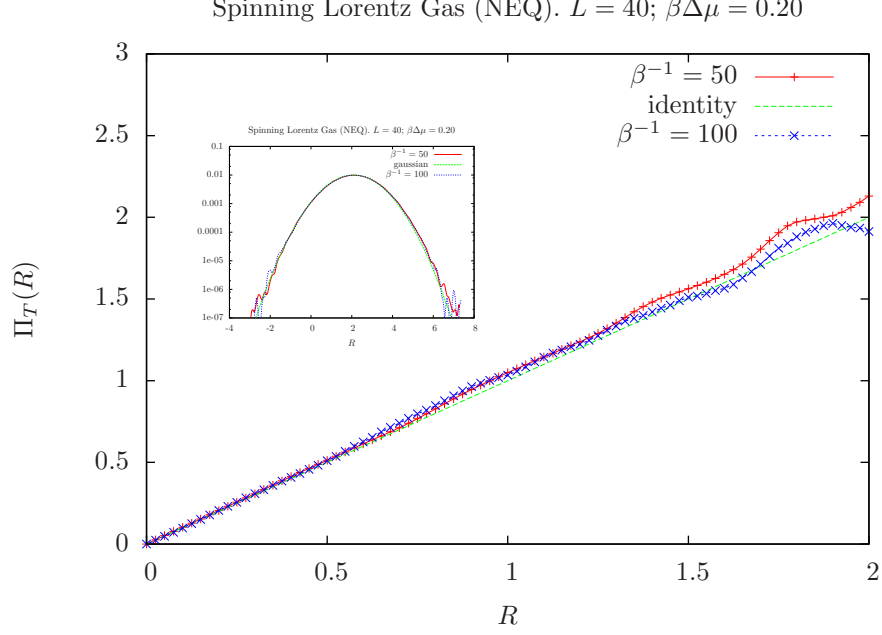


Figure 2: The fluctuation symmetry for the dynamical activity is tested numerically in nonequilibrium simulations of the SLG. In the inset, the probability distribution  $P_T(R)$  of  $R$  in equation (5.1) is given. In the vertical axis we have the calculation of the functional 5.2. The slab length is  $L = 40$ , with reservoir chemical potential difference  $\beta\Delta\mu = 0.2$ , and reservoir temperatures  $\beta^{-1} = 50$  (crosses) and  $\beta^{-1} = 100$  (stars) giving identical result.

We now wish to conjecture that identical fluctuation relations as (2.7)–(2.10) hold for the dynamical activity as we had it for the boundary driven zero range process before, in particular in the version (4.13). One therefore looks back at the expression (4.12). More precisely, we take the time-symmetric variable

$$R = \frac{\beta\Delta\mu}{2T} ((I_{\leftarrow}^{\ell} + I_{\rightarrow}^{\ell}) - (I_{\leftarrow}^r + I_{\rightarrow}^r)) \quad (5.1)$$

Figures 2 and 3 show the validity of the time-symmetric fluctuation theorem for the probability  $P_T(R)$  in the SLG model. This distribution is obtained via molecular dynamics simulations of the system in a nonequilibrium stationary state, with different reservoir chemical potentials. In these figures we plot the functional

$$\Pi_T(R) = \frac{1}{T} \ln \frac{P_T(R)}{P_T(-R)} \quad (5.2)$$

The measuring time was a large value of  $T = 4.0$ , which means that in 5.2 one understands the fluctuation symmetry in the asymptotic sense; in the same time



units, the average time between collisions in the gas is  $\sim 2.5 \times 10^{-3}$ . In the first case (Fig. 2) stationary nonequilibrium is obtained by a chemical potential difference  $\beta\Delta\mu = 0.20$  and for two different temperatures.

The second case (Fig. 3) corresponds to a larger driving  $\beta\Delta\mu = -0.45$ ; this gives a fluctuation theorem interval in which the distribution is visibly non-Gaussian.

As in the remark around (4.7), the dynamical activity in (5.1) is proportional to the number of transitions at the wall; in other words, it is proportional to the local boundary density. Since the temperature in this case is uniform, the activity fluctuations are simply related to density fluctuations in the stationary profiles. Thus, when measuring the differences in dynamical activity in (5.1) one obtains asymmetric statistics due to the asymmetry in the nonequilibrium density *profile* along the slab in fig. 1, by the condition set in the reservoirs. Thus, one observes that the variable in (5.1) is sensitive to the breaking of spatial symmetry, which is another valid aspect of nonequilibrium; this is related but also complementary to the breaking of time-reversal symmetry. That is then the physical meaning of the involution  $\Gamma$  introduced in Section 2, as previously noted in [18].

## 6 Conclusions

Fluctuation relations in nonequilibrium extend to the realm of time-symmetric observables. These observables are however antisymmetric for other transformations such as spatial symmetries, the breaking of which is responsible for the time-reversal breaking. The inhomogeneity in dynamical activity exactly picks up that (other) symmetry breaking. In particular, we have checked via computer simulations that there is a fluctuation symmetry for the difference in dynamical activity at the boundaries of the spinning Lorentz gas in stationary nonequilibrium. These were guessed from a mathematical analysis of the boundary driven Kawasaki and zero range process. We would like to call these *active* fluctuation symmetries as they do not involve the dissipative currents, but rather that complementary aspect of frenesy or dynamical activity. As we have seen, the logic and mathematical derivation of these additional time-symmetric fluctuation symmetries is completely analogous to the existing fluctuation relations for dissipated work and entropy fluxes, except that an additional symmetry is involved; most simply a mirror or reflection symmetry, which basically is equivalent to reversing the driving field. The very fact that reversing time is not equivalent with reversing the driving field is responsible for non-trivial nonequilibrium behavior beyond the linear regime.

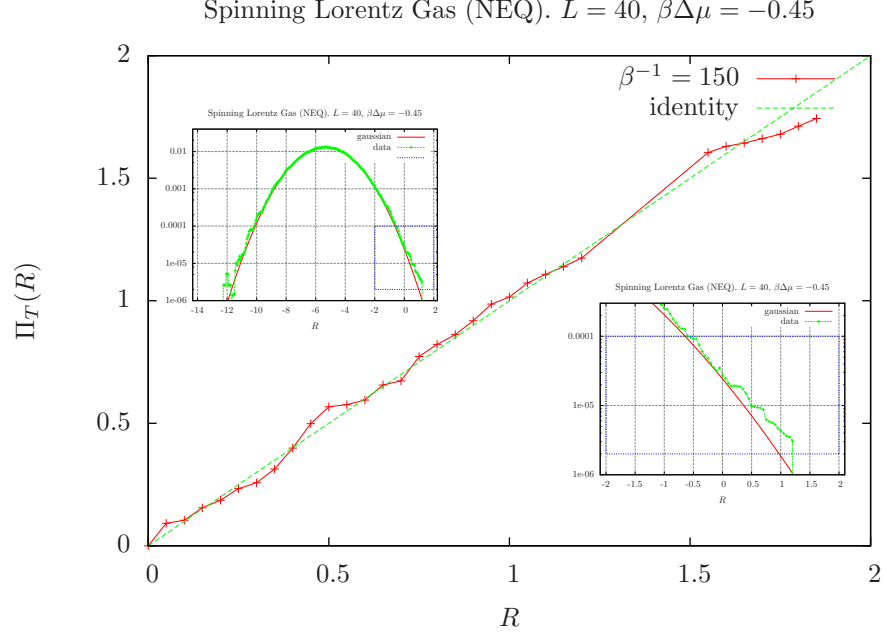


Figure 3: The validation of the time-symmetric fluctuation theorem for the case of non-Gaussian fluctuations of the dynamical activity difference  $R$ , in the SLG in stationary nonequilibrium. The chemical potential difference in the reservoirs is  $\beta\Delta\mu = -0.45$ , with  $\beta = 1/150$  and slab length  $L = 40$ . The inset shows the probability  $P_T(R)$ , measured from the computer simulation with a large measuring time  $T = 4.0$ . The interval of fluctuations around zero is far from the average value, where one distinguishes non-Gaussian behavior. In the main plot, the crosses show the evaluation of the fluctuation theorem for the probabilities in the inset; these data fit to a straight line with slope close to one,  $m = 0.99735 \pm 0.01265$ .

## References

- [1] D.J. Evans, E. G. D. Cohen and G.P. Morris, Probability of second law violations in steady flows. *Phys. Rev. Lett.* **71**, 2401–2404 (1993).
- [2] G. Gallavotti and E. G. D. Cohen, *Phys. Rev. Lett.* **74**, 2694 (1995); *J. Stat. Phys.* **80**, 931 (1995).
- [3] C. Maes, The fluctuation theorem as a Gibbs property. *J. Stat. Phys.* **95**, 367–392 (1999).
- [4] J.L. Lebowitz and H. Spohn, The Gallavotti–Cohen fluctuation theorem for stochastic dynamics. *J. Stat. Phys.* **95**, 333 (1999).
- [5] G. E. Crooks, Entropy production fluctuation theorem and the nonequilibrium work relation for free energy differences. *Phys. Rev. E* **60**, 2721–2726 (1999).
- [6] C. Maes and K. Netočný, Time-reversal and Entropy. *J. Stat. Phys.* **110**, 269–310 (2003).
- [7] C. Maes, On the origin and the use of fluctuation relations for the entropy. *Séminaire Poincaré* **2**, 29–62 Eds. J. Dalibard, B. Duplantier and V. Rivasseau, Birkhäuser (Basel), 2003.
- [8] P.G. Bergman and J.L. Lebowitz, New Approach to Nonequilibrium Process. *Phys. Rev.* **99**, 578–587, 1955.
- [9] S. Katz, J.L. Lebowitz, and H. Spohn, Stationary nonequilibrium states for stochastic lattice gas models of ionic superconductors. *J. Stat. Phys.* **34**, 497–538 (1984).
- [10] T. Harada and S.-Y. Sasa: Equality connecting energy dissipation with violation of fluctuation-response relation, *Phys. Rev. Lett.* **95**, 130602 (2005).
- [11] B. Derrida, Non-equilibrium steady states: fluctuations and large deviations of the density and of the current. *J. Stat. Mech.* P07023 (2007).
- [12] H. Tasaki, Two theorems that relate discrete stochastic processes to microscopic mechanics. arXiv:0706.1032v1 [cond-mat.stat-mech].
- [13] C. Jarzynski, *Phys. Rev. Lett.* **78**, 2690 (1997); *Phys. Rev. E* **56**, 5018 (1997).
- [14] M. Baiesi, E. Boksenbojm, C. Maes and B. Wynants, Nonequilibrium linear response for Markov dynamics, I: jump processes and overdamped diffusions, *J. Stat. Phys.* **137**, 1094–1116 (2009).
- [15] J. P. Garrahan, R.L. Jack, V. Lecomte, E. Pitard, K. van Duijvendijk, and F. van Wijland: First-order dynamical phase transition in models of glasses: an approach based on ensembles of histories, *J. Phys. A: Math. Gen.* **42**, 075007 (2009).

- [16] R. Jack, J.P. Garrahan, D. Chandler: Space-time thermodynamics and subsystem observables in kinetically constrained models of glassy materials, *J. Chem. Phys.* **125**, 184509 (2006).
- [17] F. Bonetto, G. Gallavotti, and P. Garrido, Chaotic principle: An experimental test. *Physica D* **105**, 226 (1997).
- [18] C. Maes and M.H. van Wieren, Time-symmetric fluctuations in nonequilibrium systems. *Phys. Rev. Lett.* **96**, 240601 (2006).
- [19] W. De Roeck and C. Maes, Symmetries of the ratchet current. *Phys. Rev. E* **76**, 051117 (2007).
- [20] C. Maes, K. Netočný and M. Verschuere, Heat Conduction Networks. *J. Stat. Phys.* **111**, 1219–1244 (2003).
- [21] J.W. Silverstein and P.L. Combettes, Signal detection via spectral theory of large dimensional random matrices. *IEEE Transactions on Signal Processing* **40**(8), 2100–2105 (1992).
- [22] M. Colangeli, C. Maes and B. Wynants, A meaningful expansion around detailed balance. *J. Phys. A: Math. Theor.* **44**, 095001 (2011).
- [23] One should have in mind that  $\Theta$  is time-reversal on trajectories and  $\Gamma$  is a spatial transformation. In many cases, the breaking of time-symmetry is directly related to the breaking of a spatial symmetry.
- [24] K. Sekimoto, *Stochastic Energetics*. Lecture Notes in Physics **799**, Springer (2010).
- [25] F. Ritort, C. Bustamente and I. Tinoco, Jr., A two-state kinetic model for the unfolding of single molecules by mechanical force. *Proc. Nat. Ac. Scienc.* **99**, 13544-13538 (2002).
- [26] J. Kurchan, Fluctuation theorem for stochastic dynamics. *J. Phys. A: Math. Gen.* **31**, 3719-3729 (1998).
- [27] C. Maes, F. Redig and A. Van Moffaert, On the definition of entropy production, via examples. *J. Math. Phys.* **41**, 1528-1554 (2000).
- [28] D. Ruelle, Smooth Dynamics and New Theoretical Ideas in Nonequilibrium Statistical Mechanics. *J. Stat. Phys.* **95**, 393-468 (1999).
- [29] C. Maes and E. Verbitskiy, Large deviations and a fluctuation symmetry for chaotic homeomorphisms. *Comm. Math. Phys.* **233**, 137–151 (2003).
- [30] D. Andrieux and P. Gaspard, The fluctuation theorem for currents in semi-Markov processes. *J. Stat. Mech.:Theory and Experiment*, P11007 (2008).
- [31] C. Maes, K. Netočný and B. Wynants, Dynamical fluctuations for semi-Markov processes. *J. Phys. A: Math. Theor.* **42**, 365002 (2009).

- [32] G. Gallavotti, Extension of Onsagers Reciprocity to Large Fields and the Chaotic Hypothesis. *Phys. Rev. Lett.* **77**, 4334–4337 (1996).
- [33] R. Kubo, The fluctuation-dissipation theorem. *Rep. Prog. Phys.* **29**, 255–284 (1966).
- [34] T. S. Komatsu and N. Nakagawa, An expression for stationary distribution in nonequilibrium steady states. *Phys. Rev. Lett.* **100**, 030601 (2008).
- [35] C. Maes and K. Netočný, Rigorous meaning of McLennan ensembles. *J. Math. Phys.* **51**, 015219 (2010).
- [36] C. Maes, K. Netočný, and B. Wynants, On and beyond entropy production; the case of Markov jump processes. *Markov Proc. Rel. Fields.* **14**, 445–464 (2008).
- [37] C. Maes, K. Netočný and B. Shergelashvili, A selection of nonequilibrium issues. In: *Methods of Contemporary Mathematical Statistical Physics*, Ed. Roman Kotecký, *Lecture Notes in Mathematics* 1970, pp. 247–306, Springer, 2009.
- [38] R. J. Harris, A. Rákos and G.M.Schütz, Breakdown of Gallavotti-Cohen symmetry for stochastic dynamics. *Europhys. Lett.* **75**, 227 (2006).
- [39] H. A. Lorentz, *Proc. Amst. Acad.* **438** (1905).
- [40] P. Gaspard, *Chaos, Scattering, and Statistical Mechanics* (Cambridge University Press, Cambridge, 1998).
- [41] *Hard Ball Systems and the Lorentz Gas*, edited by D. Szász (Springer Verlag, Berlin, 2000).
- [42] H. Larralde, F. Leyvraz and C. MejaMonasterio, Transport properties of a modified Lorentz gas. *J. Stat. Phys.* **113**, 197–231 (2003).
- [43] A. Salazar, F. Leyvraz and H. Larralde, Fluctuation theorem for currents in the Spinning Lorentz Gas. *Physica A* **388**, 4679–4694 (2009).
- [44] F. Reif, *Fundamentals of Statistical and Thermal Physics* (International Student Edition), McGraw-Hill, Tokyo, 1965; pp. 272.